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MINIMUM-WEIGHT MULTI-CONSTRAINT VIBRATING CANTILEVERS

B. L. KARIHALOO and R. D. PARBERY†

Department of Civil Engineering, The University of Newcastle, New South Wales 2308, Australia

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Abstract—The paper considers the problem of minimising the mass of vibrating cantilevers whose fundamental frequencies of natural vibrations in longitudinal and transverse modes exceed prescribed values. The cantilevers are supposed to perform longitudinal and transverse harmonic vibrations at different times during their design life. Solutions are presented for members whose cross-section is of solid construction. It is shown that the optimally designed members are substantially lighter than the corresponding prismatic members.

INTRODUCTION

Eigenvalue problems relating to the dynamic behaviour of one- and two-dimensional structural/mechanical elements have prominently figured in the optimal design of such elements. By optimal is meant the design that acheives certain prescribed frequencies of natural vibrations by using the least amount of material. Optimal design problems concerning small longitudinal and transverse harmonic vibrations of strings, beams and plates have received considerable attention over the years. Thus, Schwarz[1, 2] studies the extrema of natural frequencies of nonhomogeneous strings and bars. The first big step in the optimal design of transversely vibrating beams was taken by Niordson[3] and of longitudinally vibrating bars—by Turner[4]. Niordson's pioneering work has since been extended to other elements (see, e.g. [5–8]).

Mathematically, the optimization problems reduced to the solution of an isoperimetric variational problem with a single constraint. However, it is often interesting to design rather versatile structures that are able to fulfil several functions at different times during their design life (multi-purpose structures). Some simple examples of multi-purpose optimal designs were given by Prager and Shield[9], and Martin[10]. Of late, considerable attention has been paid to this optimal design of multi-purpose structures (see, e.g. [11–15]).

The present paper also addresses itself to a multi-purpose design situation. It considers the problem of minimising the mass of vibrating cantilevers whose fundamental frequencies of harmonic natural vibrations in longitudinal and transverse modes exceed certain prescribed minimum values. The cantilevers are supposed to perform longitudinal and transverse harmonic vibrations at different times during their design life. Solutions are presented for members whose cross-section is of solid construction. It is shown that optimization can lead to significantly lighter designs.

MATHEMATICAL FORMULATION

We consider the problem of minimising the volume (mass) of a cantilever beam with a specified non-structural mass at its tip (Fig. 1). The beam performs longitudinal and transverse vibrations at different times during its design life. The design requirement is that the fundamental frequencies in longitudinal and transverse vibration modes be greater than specified values.

Small, harmonic longitudinal vibrations of fundamental frequency Ω are described by the following eigenvalue problem in a non-dimensional form:

$$(\alpha_1 u_x)_x + \beta\alpha_1 u = 0 \tag{1}$$

$$u(0) = 0 \tag{2}$$

$$(\alpha_1 u_x)|_{x=1} = M\beta u|_{x=1}, \tag{3}$$

where $u(x)$ is the non-dimensional longitudinal displacement amplitude, $\alpha_1(x) = A(x)/L^2$, $M =$

†Department of Mechanical Engineering.

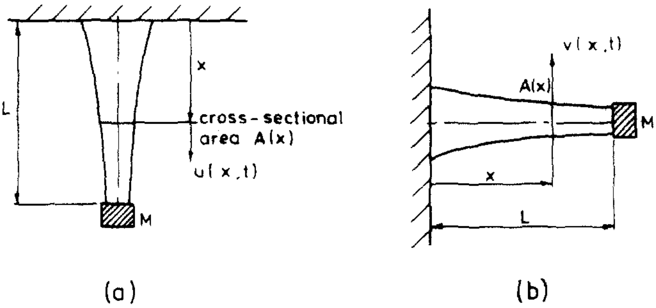

Fig. 1. Cantilever with added non-structural mass, in (a) longitudinal mode, (b) transverse mode.

$M/\rho L^3$, $\beta = \Omega^2 \rho L^2/E$, and ρ is the density of the beam material, $A(x)$ the cross-sectional area and E Young's modulus. The linear dimensions have been non-dimensionalised by L, and subscript x denotes differentiation with respect to the non-dimensional longitudinal co-ordinate x.

Small, harmonic transverse vibrations of fundamental frequency ω are described by the following eigenvalue problem in a non-dimensional form:

$$(\alpha_1{}^n v_{xx})_{xx} - \gamma \alpha_1 n = 0 \tag{4}$$

$$v(0) = v_x(0) = 0 \tag{5}$$

$$(\alpha_1{}^n v_{xx})|_{x=1} = 0 \tag{6}$$

$$(\alpha_1{}^n v_{xx})_x|_{x=1} = -M\lambda\, v|_{x=1}, \tag{7}$$

where $v(x)$ is the non-dimensional transverse displacement amplitude and $\lambda = \omega^2 \rho L^{(6-2n)}/Ec$. Here, it is assumed that the second moment of area $I(x)$ and the cross-sectional area $A(x)$ are related through

$$I(x) = c\,A^n(x), \tag{8}$$

where c and n are constants determined by the cross-sectional shape.

Mathematically, the optimization problem under consideration reduces to

$$\min_{\alpha(x)} \int_0^1 \alpha(x)\,dx, \tag{9}$$

$$\frac{\lambda}{M^{(n-1)}} = \frac{\int_0^1 \alpha^n v_{xx}^2\,dx}{\int_0^1 \alpha v^2\,dx + 1} \geq A, \tag{10}$$

$$\beta = \frac{\int_0^1 \alpha u_x^2\,dx}{\int_0^1 \alpha u^2\,dx + 1} \geq B, \tag{11}$$

where A and B are prescribed positive constants and $\alpha(x) = \alpha_1(x)/M$. As $u(x)$ and $v(x)$ are determined only to within a constant multiplier, they have been normalised so that $u^2(1) = v^2(1) = 1$. Expressions (10) and (11) are the familiar Rayleigh quotients.

OPTIMALITY CONDITION

In order to derive the necessary optimality condition for the above optimisation problem we form an auxiliary functional

$$\Pi = \int_0^1 \alpha\, dx + \nu\left[(A+s^2)\left(\int_0^1 \alpha\, v^2\, dx + 1\right) - \int_0^1 \alpha^n\, v_{xx}^2\, dx\right]$$
$$+ \mu\left[(B+g^2)\left(\int_0^1 \alpha\, u^2\, dx + 1\right) - \int_0^1 \alpha\, u_x^2\, dx\right], \quad (12)$$

where ν and μ are (constant) Lagrange multipliers and s^2 and g^2 are positive slack variables.

$\operatorname{Min}_{\alpha} \int_0^1 \alpha\, dx$ is achieved by minimising Π with respect to α, s and g. Considering the variations in Π due to s and g

$$\frac{\delta \Pi}{\delta s} = 2\nu s\left(\int_0^1 \alpha\, v^2\, dx + 1\right) \quad (13)$$

$$\frac{\delta \Pi}{\delta g} = 2\mu g\left(\int_0^1 \alpha u^2\, ds + 1)\right) \quad (14)$$

and noting that the bracketed terms are positive, we obtain that, for $\mu > 0$, $\nu > 0$, $s = g = 0$ for min Π. The cases where either $\mu = 0$ or $\nu = 0$ will be considered separately.

From the stationary condition for Π with respect to α we get

$$\int_0^1 [1 + \nu(Av^2 - n\alpha^{n-1}v_{xx}^2) + \mu(Bu^2 - u_x^2)]\delta\alpha\, dx = 0 \quad (15)$$

Recognising that $\delta\alpha$ is an arbitrary function, we obtain the following optimality condition:

$$\nu(n\alpha^{n-1}v_{xx}^2 - Av^2) + \mu(\mu_x^2 - Bu^2) = 1. \quad (16)$$

It should be noted that the condition $s = g = 0$ means that the equality sign applies in the constraints (10) and (11).

Before proceeding to discuss the solution of the multi-constraint optimisation problem it is interesting to investigate the special cases when either $\mu = 0$ or $\nu = 0$.

SPECIAL CASES

When $\nu = 0$, $s^2 > 0$, the problem is equivalent to that of minimising the volume subject to constraint (11) alone, i.e. only the constraint on fundamental frequency of longitudinal vibrations is active, the frequency of transverse vibrations being greater than that specified. The solution to this problem was reported by Turner[4].

In this case the optimality condition (16) reduces to

$$u_x^2 - Bu^2 = 1/\mu \quad (17)$$

When (17) is solved together with (1)–(3) and (11) the solution is[4]

$$u(x) = \sinh(Cx)/\sinh C \quad (18)$$

$$\alpha(x) = C \sinh C \cosh C/\cosh^2(Cx) \quad (19)$$

$$\text{Volume} = \sinh^2 C, \quad (20)$$

where $C = \sqrt{B}$.

If the optimal design for longitudinal vibrations is used in transverse vibration mode then the fundamental frequency may be found from eqns (4)–(7), where α is given by (19). The following iterative procedure was adopted:

(i) Divide the range $x \in [0, 1]$ into m equal parts.

(ii) Set $v_{xx}(x) \equiv 1$, $x \in [0, 1]$ in the first iteration.

(iii) Find $v_x(x) = \int_0^x v_{\eta\eta}\, d\eta$. (Note $v_x(0) = 0$.)

(iv) Find $v(x) = \int_o^x v_\eta\, d\eta$. (Note $v(0) = 0$.)

(v) Normalise v_{xx}, v_x and v by dividing by $v(1)$.

(vi) Find $\lambda/M^{n-1} = \int_0^1 \alpha^n v_{xx}^2\, dx/(\int_0^1 \alpha v^2\, dx + 1)$.

(vii) Find a new value of v_{xx} satisfying boundary conditions (6) and (7):

$$v_{xx} = \frac{\lambda}{M^{n-1}}\left[(1-x) + \int_x^1 d\eta \int_\eta^1 \alpha v\, d\xi\right] \Big/ \alpha^n(x).$$

(viii) Repeat Steps (iii)–(vii) until the successive values of λ/M^{n-1} differ by less than 10^{-5}. The results are shown graphically in Figs. 2 and 3.

The second special case is when $\mu = 0$, $g^2 > 0$. This corresponds to the case of a beam optimally designed for transverse vibrations only which when used in longitudinal vibration mode will have a frequency greater than that specified. The optimal shape of a cantilever beam in transverse vibrations was obtained by Karihaloo and Niordson[7]. Here the optimality condition (16) reduces to

$$n\, \alpha^{(n-1)}\, v_{xx}^2 - A\, v^2 = 1/\nu. \tag{21}$$

The solution for this case is complicated by the fact that $\alpha(1) = 0$ and a singularity occurs in v_{xx} at $x = 1$. Following [7], we put

$$v_{xx} = g(x)\,(1-x)^{(1-n)/(1+n)}, \tag{22}$$

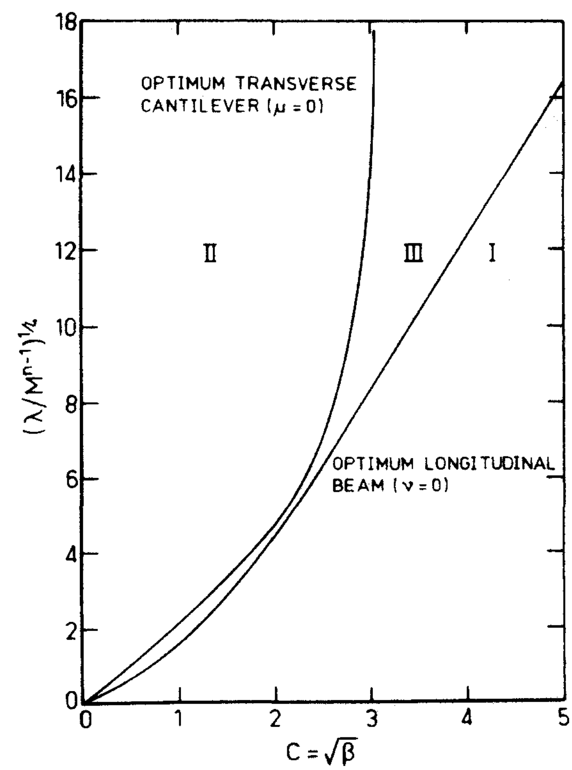

Fig. 2. Variation of β with λ/M showing the three design regions for $n = 2$.

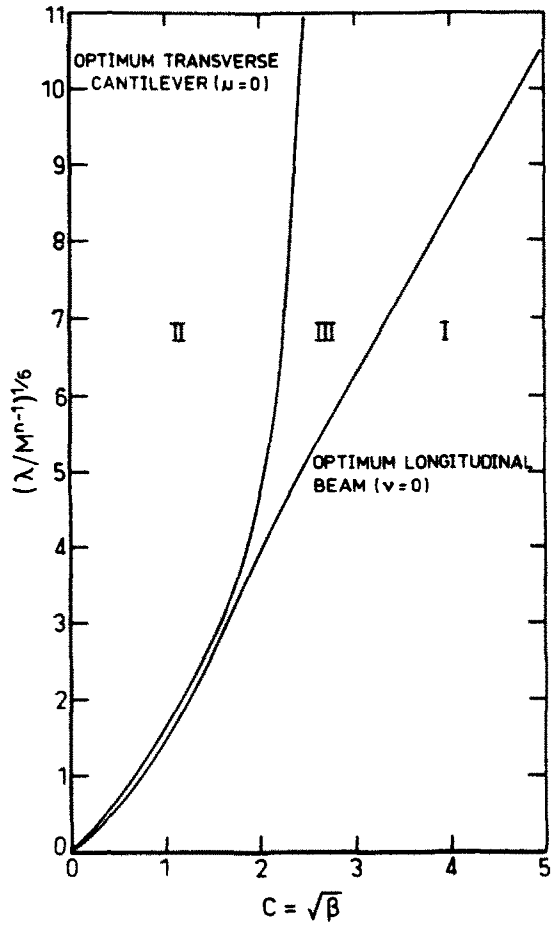

Fig. 3. Variations of β with λ/M^2 showing the three design regions for $n = 3$.

where $g(x)$ is a regular function. Substituting (22) into (21) we obtain

$$\alpha(x) = \left[\frac{1/\nu + Av^2}{ng^2(x)}\right]^{1/(n-1)} (1-x)^{2/(n+1)}. \tag{23}$$

Similarly, by substituting (22) and (23) into (4), integrating twice and satisfying the boundary condition (6), (7), we get

$$g(x) = \left[\frac{1/\nu + A\,v^2(x)}{n}\right]^{n/(n+1)} \left[\frac{(1-x)}{A\left[(1-x) + \int_x^1 \mathrm{d}\eta \int_\eta^1 \alpha\, v\, \mathrm{d}\xi\right]}\right]^{(n-1)/(n+1)}. \tag{24}$$

By applying L'Hopital's rule and recalling that $v(1) = 1$, we find

$$g(1) = \left[\frac{1/\nu + A}{n}\right]^{n/(n+1)} \Big/ A^{(n-1)/(n+1)}.$$

The iterative scheme used to solve for $\alpha(x)$ and $v(x)$ was the same as used in [7].

The calculation of the fundamental frequency of the optimally designed beam when used in longitudinal vibration mode is not straight forward. This is because the optimally designed beam exhibits $\alpha(1) = 0$ and hence from the boundary condition (3) a singularity must occur in u_x at $x = 1$.

Let us investigate the behaviour of $u(x)$ in the vicinity of $x = 1$ when $\alpha(x)$ is given by (23).

To this end we rewrite (23)

$$\alpha(x) = C(x)\,(1-x)^{2/(n+1)}, \tag{25}$$

where $C(x)$ is a regular, non-zero function.

We assume that in the neighbourhood of $x = 1$, $u(x)$ may be expanded as

$$u(x) = 1 + D(1-x)^k + \ldots\ldots, \tag{26}$$

where k is a positive non-integer number and D is a constant.

Writing the differential equation (1) as

$$\alpha_x u_x + \alpha u_{xx} + \beta\alpha\, u = 0 \tag{27}$$

and substituting from (25) and (26) for α and u, we obtain after some simplification

$$(1-x)^{2/(n+1)}[1+(1-x)^k] + \left[\frac{2k}{n+1} + k^2 - k\right](1-x)^{k-(2n/(n+1))} = 0. \tag{28}$$

As $x \to 1$, eqn (28) is dominated by the term with the lowest power of $(1-x)$; in this case $k - 2n/(n+1)$. Equating the coefficient of this term to zero, we get

$$k = 0 \quad \text{or} \quad k = (n-1)/(n+1). \tag{29}$$

The possibility $k = 0$ is excluded as $u(x)$ could not satisfy the boundary condition (3) in that case. We conclude that $k = (n-1)/(n+1)$ and that $u_x \propto (1-x)^{-2/(n+1)}$ in the vicinity of $x = 1$.

Introducing a regular function $f(x)$, we put

$$u_x = f(x)\,(1-x)^{-2/(n+1)}. \tag{30}$$

Integrating the differential equation (1) once with respect to x and using the boundary condition (3), we obtain

$$\alpha\, u_x = \beta\left[1 + \int_x^1 \alpha\, u\, \mathrm{d}\eta\right] = h(x). \tag{31}$$

Substituting (25) and (30) into (31), we get

$$f(x) = h(x)/C(x). \tag{32}$$

The fundamental frequency of longitudinal vibrations of the beam optimally designed for transverse vibrations was found as follows:

(i) Assume $f(x) \equiv 1$ in the first iteration.

(ii) Find $u(x) = \int_0^x f(\eta)(1-\eta)^{-2/(n+1)}\,\mathrm{d}\eta$.

(iii) Normalise $u(x)$ and $f(x)$ so that $u(1) = 1$.

(iv) Calculate

$$\beta = \frac{\int_0^1 \alpha\, u_x^2\,\mathrm{d}x}{\int_0^1 \alpha\, u^2\,\mathrm{d}x + 1} = \frac{\int_0^1 C(x)\,f^2(x)(1-x)^{-2/(n+1)}\,\mathrm{d}x}{\int_0^1 C(x)\,(1-x)^{2/(n+1)}\,u^2\,\mathrm{d}x + 1}.$$

(v) Calculate a new value of $f(x)$

$$f(x) = h(x)/C(x).$$

(vi) Repeat Steps (ii)–(v) if the successive values of β differ by more than 10^{-6}.
(vii) Repeat Steps (i)–(vi) for various values of A and $n = 2$ or 3.
The results are shown in Figs. 2 and 3.

MULTI-CONSTRAINT OPTIMIZATION

We now turn our attention to the solution of the problem when both frequency constraints are active. In this case it is easy to show that $\alpha(1)$ is nonzero and that no singularities occur in the solution at $x = 1$.

Integrating (4) twice and using boundary conditions (6) and (7) one obtains

$$\alpha^n(x)\, v_{xx} = f(x) = A\left[(1-x) + \int_x^1 \mathrm{d}\eta \int_\eta^1 \alpha\, v\, \mathrm{d}\xi\right]. \tag{33}$$

Similarly from (1) to (3) one gets

$$\alpha(x)\, u_x = g(x) = B\left[1 + \int_x^1 \alpha\, u\, \mathrm{d}\eta\right]. \tag{34}$$

Multiplying the optimality condition (16) by α^{n+1}, using (33), (34) and rearranging we have an implicit expression for $\alpha(x)$:

$$\alpha(x) = \left[\frac{\nu n f^2(x) + \mu g^2(x)\alpha^{n-1}}{1 + \nu A v^2 + \mu B u^2}\right]^{1/(n+1)}. \tag{35}$$

Multiplying (16) by α, integrating and using (10) and (11) with equality sign we obtain the following expression for ν

$$\nu = \frac{\int_0^1 \alpha\, \mathrm{d}x - \mu B}{A\left[(n-1)\int_0^1 \alpha\, v^2\, \mathrm{d}x + n\right]}. \tag{36}$$

The following procedure was used:

(i) Prescribe values of A and B, and estimate μ in the range $0 < \mu < \mu_l$ where μ_l is the value of μ corresponding to the optimal longitudinally vibrating beam which has the prescribed value of A.
(ii) In the first iteration set $v_{xx} = 1$ and $u_x = 1$.
(iii) Calculate u, v_x, v

$$u = \int_0^x u_\eta\, \mathrm{d}\eta$$
$$v_x = \int_0^x v_{\eta\eta}\, \mathrm{d}\eta$$
$$v = \int_0^x v_\eta\, \mathrm{d}\eta.$$

(iv) Normalise u, v so that $u(1) = v(1) = 1$.
(v) In the first iteration put $\alpha(x) = 1$. Estimate ν, β.
(vi) Calculate $f(x)$ from (33).
(vii) Calculate $g(x)$ from (34).
(viii) Calculate $\alpha(x)$ using (35).
(ix) Calculate ν from (36).

(x) Calculate

$$\beta = \frac{\int_0^1 \alpha\, u^2\, dx + 1}{\int_0^1 \frac{g^2(x)}{\alpha(x)}\, dx}.$$

(xi) Repeat Steps (vi)–(x) until the difference between successive values of both ν and β is less than 0.001%.

(xii) For the next iteration calculate new values of u_x and v_{xx}:

$$u_x = \beta\left[1 + \int_x^1 \alpha\, u\, d\eta\right] \Big/ \alpha(x)$$

$$v_{xx} = A\left[(1-x) + \int_x^1 \int_\eta^1 \alpha\, v\, d\xi\, d\eta\right] \Big/ \alpha^n(x).$$

(xiii) Repeat Steps (iii)–(xii) until the difference between successive values of both $u_x(0)$ and $v_{xx}(0)$ is less than 0.001%.

(xiv) In general the value of β calculated in Step (x) is not equal to the required value B. Repeat (ii)–(xiii) with different values of ν, using Newton's method on μ and β to obtain

$$\left|\frac{B-\beta}{B}\right| < 10^{-5}.$$

(xv) Repeat Steps (ii)–(xiv) for various combinations of A and B as required.

The scheme outlined should not be used at combinations of A and B too close to those corresponding to the limiting case with only the transverse frequency constraint acitve, because it can not cope with the singularities present in the limiting case.

RESULTS

Figures 2 and 3 show the regions of interest in the plane of the constraint variables for $n = 2$ and 3 respectively. Region I represents the region where the fundamental frequency of longitudinal vibrations alone controls the design; Region II is the region where the fundamental frequency of transverse vibrations alone controls the design; and Region III is the region where both constraints are active.

It may be noted that when both frequencies are small the region where both constraints are active is rather narrow so that savings compared to the single purpose designs might be expected to be small.

However, when both frequencies are high the region where both constraints are active is wide and substantial savings may be expected.

Figures 4 and 5 show the values of μ and ν for the limiting cases of the optimally designed transversely vibrating beam and the optimally designed longitudinally vibrating beam respectively. The range of μ and ν in Region III is from 0 to the value shown on the graph in each case.

Figures 6 and 7 show some examples of the optimum shape. Figure 6 shows the variation of $(\alpha_1/M)^{1/2}$ (linear dimension) with x for various values of λ/M and β when $n = 2$. Figure 7 shows the variation of α_1/M for selected values of λ/M^2 and β when $n = 3$. It will be recalled that for $n = 2$ the diameter is proportional to $\alpha^{1/2}$ and for $n = 3$ the depth is proportional to α.

CONCLUSION

In order to judge the savings due to the optimisation it is instructive to compare the volume of the optimal beam with that of a uniform prismatic beam satisfying the same design constraints.

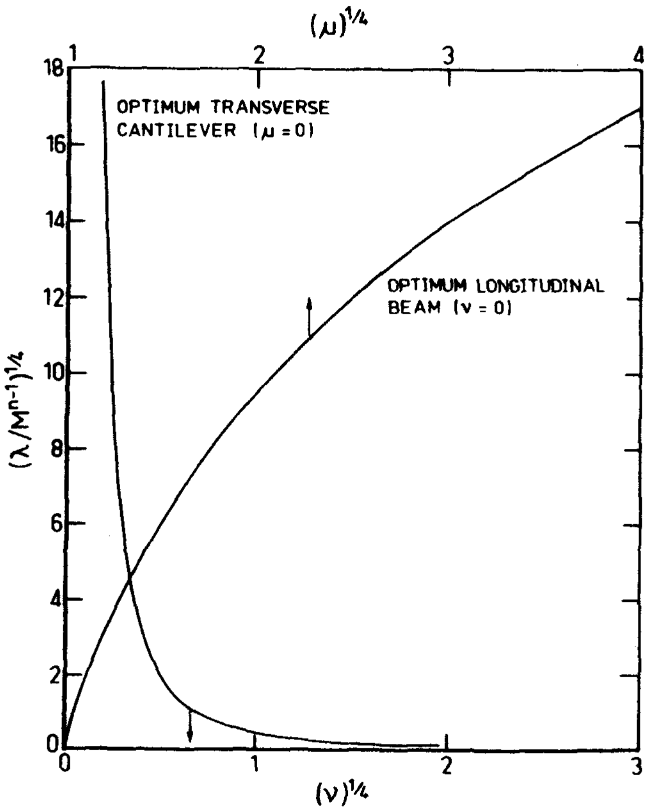

Fig. 4. Variation of Lagrange multipliers μ and ν with λ/M^{n-1} for extreme cases. $n = 2$.

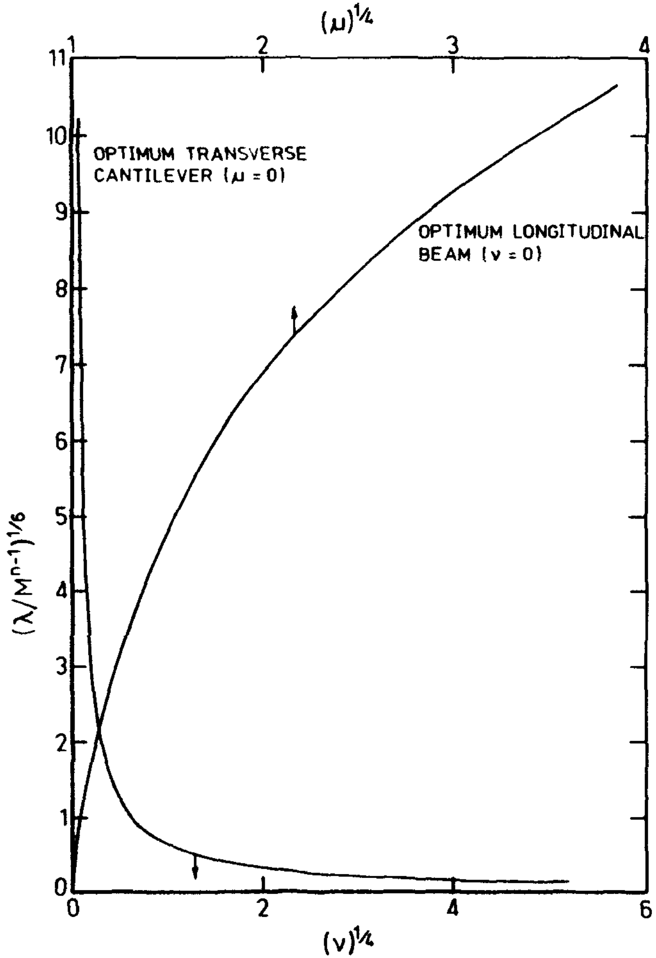

Fig. 5. Variation of Lagrange multipliers μ and ν with λ/M^{n-1} for extreme cases. $n = 3$.

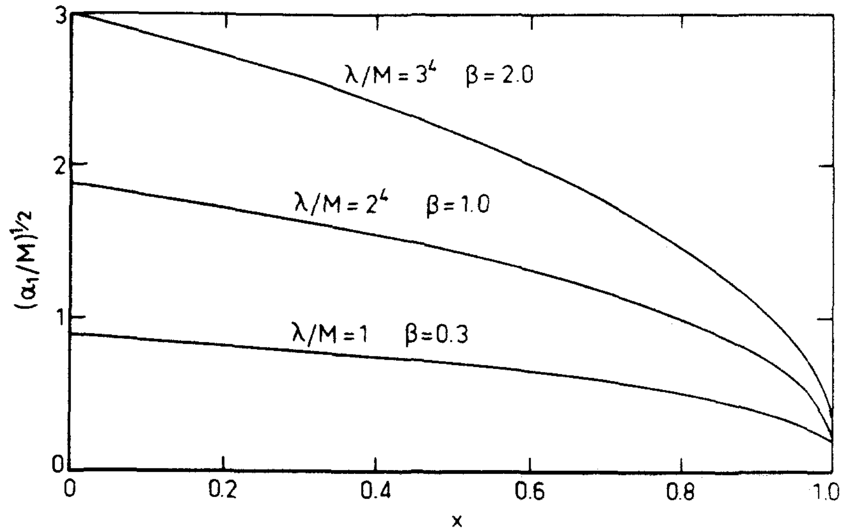

Fig. 6. Shapes of the optimal beam ($\alpha^{1/2}$) for various values of β, λ/M; $n = 2$.

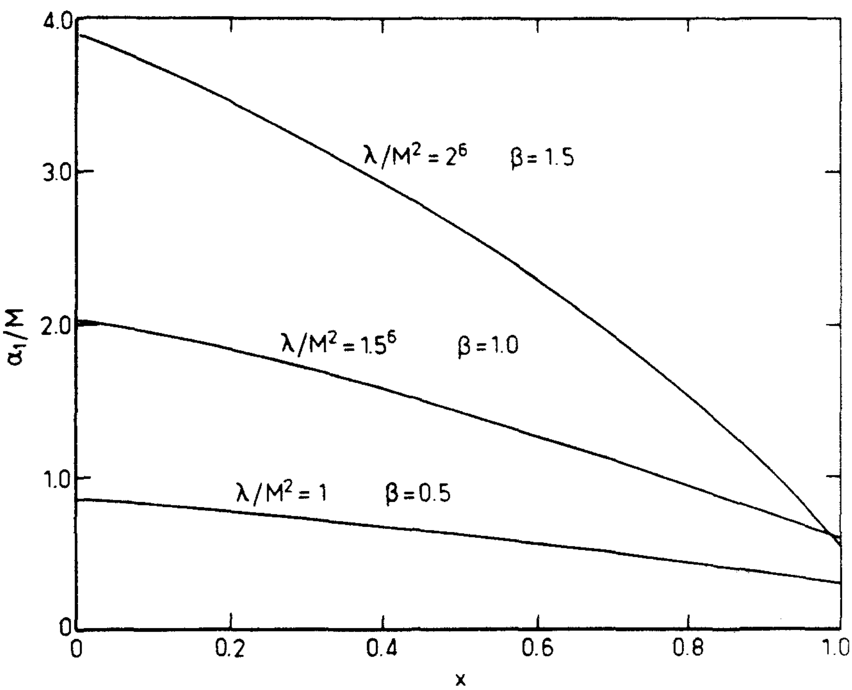

Fig. 7. Shapes of the optimal beam (α) for various values of β, λ/M^2; $n = 3$.

With α_1 constant eqn (1) reduces to

$$u_{xx} + \beta u = 0 \tag{37}$$

which, when boundary condition (2) is taken into account, has a solution

$$u = D\sqrt{\beta}\ \sin\ \sqrt{\beta}x \tag{38}$$

where D is constant. Substituting (37) into boundary condition (3) and rearranging gives

$$\alpha = \frac{\alpha_1}{M} = \sqrt{\beta}\ \tan\ \sqrt{\beta}. \tag{39}$$

With α_1 constant eqn (4) reduces to

$$v_{xxxx} - \frac{\lambda}{\alpha_1^{n-1}} v = 0. \tag{40}$$

The solution of eqn (40) with boundary conditions (5) and (6) is

$$v = F\left[\cos cx - \cosh cx + \left(\frac{\cos c + \cosh c}{\sin c + \sinh c}\right)(\sinh cx - \sin cx)\right]$$

where

$$c^4 = \frac{\lambda}{\alpha_1^{n-1}} = \left(\frac{\lambda}{M^{n-1}}\right)\left(\frac{M}{\alpha_1}\right)^{n-1} \tag{41}$$

and F is a constant. Substituting into boundary condition (7) and rearranging gives

$$\frac{\alpha_1}{M} = \alpha = \frac{c(\sin c \cosh c - \cos c \sinh c)}{\cos c \cosh c + 1}. \tag{42}$$

For given values of β and λ/M^{n-1} the value of α for the prismatic beam is the greater of the values given by eqns (39) and (42).

The savings compared to a prismatic design have been calculated for the examples shown in Figs. 6 and 7 and are shown in Table 1 where V_{opt} signifies the volume of the optimally designed beam and V_{prism} that of the prismatic beam satisfying the same design constraints.

It may be concluded that the savings due to the optimisation are significant and increase as the frequencies increase. Furthermore, as may be noted from eqn (39), the prismatic beam cannot satisfy the design requirements at all if the value of β exceeds $\pi^2/4$, whereas it is still possible to provide an optimally designed beam.

Table 1. Savings due to optimisation

		$n = 2$
$\frac{\lambda}{M}$	β	$\frac{V_{prism} - V_{opt}}{V_{prism}} \times 100\%$
1^4	0.30	23%
2^4	1.00	34%
3^4	2.00	46%
		$n = 3$
$\frac{\lambda}{M^2}$	β	$\frac{V_{prism} - V_{opt}}{V_{prism}} \times 100\%$
1^6	0.50	17%
1.5^6	1.00	20%
2^6	1.50	27%

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